Math 246C Lecture 24 Notes

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1 General Hörmander's Theorem and Application to Interpolation by Holomorphic Functions

1.1 Hörmander's theorem for arbitrary subharmonic functions

Theorem 1.1. *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $\varphi \in \mathrm{SH}(\Omega)$ with $\varphi \not\equiv -\infty$. Let $a > 0$, and assume that $f \in L^2_{\mathrm{loc}}$ is such that*

$$\int |f|^2 e^{-\varphi} (1 + |z|^2)^{2-a} < \infty.$$

Then there exists a u solving $\overline{\partial} u = f$ such that

$$a \int_\Omega |u|^2 e^{-\varphi} (1 + |z|^2)^{-a} \leq \int |f|^2 e^{-\varphi} (1 + |z|^2)^{2-a}.$$

Proof. This estimate has been proved if $\varphi \in C^\infty$. In general, let $\Omega_j \subseteq \Omega$ be open, relatively compact, and increasing to Ω, and let $\varphi_j \in C^\infty(\Omega_j) \cap \mathrm{SH}(\Omega_j)$ such that $\varphi_j \downarrow \varphi$. Then

$$\int_\Omega |f|^2 e^{-\varphi_j} (1 + |z|^2)^{2-a} \leq \int_\Omega |f|^2 e^{-\varphi} (1 + |z|^2)^{2-a} \leq C \qquad \forall j.$$

We get that there exists some u_j solving $\overline{\partial} u_j = f$ in Ω_j such that

$$\int_{\Omega_j} |u_j|^2 e^{-\varphi_j} (1 + |z|^2)^{-a} \leq C, \qquad j = 1, 2, \dots.$$

Let j be fixed, and consider $(u_j)_{j=k}^\infty$ on Ω_k:

$$\int_{\Omega_k} |u_j|^2 e^{-\varphi_k} (1 + |z|^2)^{-a} \leq \int_{\Omega_j} |u_j|^2 e^{-\varphi_j} (1 + |z|^2)^{-a} \leq C.$$

So $(u_j)_{j=k}^\infty$ is bounded in $L^2(\Omega_k, e^{-\varphi_k})$.

Extracting a weakly convergent subsequence and using a diagonal argument, we get a subsequence (u_{j_ν}) and $u \in L^2_{\mathrm{loc}}(\Omega)$ such that $u_{j_\nu} \to u$ weakly in $L^2(\Omega_k, e^{-\varphi_k})$ for all k. Then $\overline{\partial} u = f$ in Ω: for any $\beta \in C_0^\infty(\Omega_k)$, $\int u_{j_\nu}\beta \to \int u\beta$, so $\overline{\partial} u_{j_\nu} = f$ on Ω_k for large ν. We have $-\int u_{j_\nu}\overline{\partial}\beta = \int f\beta$ and thus $\overline{\partial} u = f$ on Ω_K.

To get the bound for u, recall that if H is a Hilbert space and $x_j \to x$ weakly in H, then $\|x\| \le \liminf_j \|x_j\|$. We get that for any k,

$$\begin{aligned} a\int_{\Omega_k} |u|^2 e^{-\varphi_k}(1+|z|^2)^{-a} &\le \liminf_{\nu\to\infty} \int_{\Omega_k} |u_{j_\nu}|^2 e^{-\varphi_k}(1+|z|^2)^{-a} \\ &\le \int_\Omega |f|^2 e^{-\varphi}(1+|z|^2)^{2-a}. \end{aligned}$$

Let $k \to \infty$ and use the monotone convergence theorem to get

$$\int_\Omega |u|^2 e^{-\varphi}(1+|z|^2)^{-a} \le \int_\Omega |f|^2 e^{-\varphi}(1+|z|^2)^{2-a}. \qquad \square$$

1.2 Application: Interpolation by holomorphic functions

Here is an application of Hörmander's theorem.

Proposition 1.1. *Let $(b_k)_{k=-\infty}^{\infty}$ be a bounded sequence in $\mathbb{C}$. There exists an $h \in \mathrm{Hol}(\mathbb{C})$ with suitable growth properties such that $h(k) = b_k$ for every $k \in \mathbb{Z}$.*

Proof. Let us first find a C^∞ solution: let $\psi \in C_0^\infty(\mathbb{C})$ be such that

$$\psi(z) = \begin{cases} 1 & |z| \le 1/4 \\ 0 & |z| \ge 1/3. \end{cases}$$

Then $g(z) = \sum_{k\in\mathbb{Z}} b_k\psi(z-k)$ is locally finite and solves the problem. We have $g \in (C^\infty \cap L^\infty)(\mathbb{C})$. Try to construct $h \in \mathrm{Hol}(\mathbb{C})$ in the form $h = g - u$, where $0 = \overline{\partial} h = \overline{\partial} g - \overline{\partial} u$. The function h will only satisfy the equation in the weak sense, but by Weyl's lemma (proved on homework last quarter), this will give $h \in \mathrm{Hol}(\mathbb{C})$ since $h \in C^\infty$.

We will also need $u|_{\mathbb{Z}} = 0$. Solve $\overline{\partial} u = \overline{\partial} g$. If we can solve this equation, then since $\overline{\partial} g \in C^\infty$, we get that $u \in C^\infty(\mathbb{C})$ by Weyl's lemma. By Hörmander's theorem for any $\varphi \in \mathrm{SH}(\mathbb{C})$, there is a solution u such that

$$a\int |u|^2 e^{-\varphi}(1+|z|^2)^{-a} \le \int |\overline{\partial} g|^2 e^{-\varphi}(1+|z|^2)^{2-a} < \infty.$$

Idea (due to Bombieri[1]): choose φ such that $\varphi|_{\mathbb{Z}} = -\infty$ and $e^{-\varphi} \notin L^1$ near $z = k$ for all k, while the right hand side is finite. This will imply that $u(k) = 0$ for all $k \in \mathbb{Z}$. Try:

$$\varphi(z) = 2\log|\sin(\pi z)| + \log(1+|z|^2).$$

[1] This idea came some time after Hörmander's theorem. It was originally for the several complex variable case, but we can use it in this case with no issue.

Then

$$e^{-\varphi} \sim \frac{1}{|z-k|^2} \notin L^1 \text{ near } z = k$$

Also take $a = 2$. Check that the right hand side equals

$$\int |\overline{\partial} g|^2 \frac{1}{|\sin(\pi z)|^2} \frac{1}{1+|z|^2} \, L(dz).$$

Since $\overline{\partial} g = \sum b_k \overline{\partial} \psi(z-k)$, $1/|\sin(\pi z)|$ is bounded on the support of $\overline{\partial} g$.

We get that $h = g - u$, which is a holoorphic solution of $h(k) = b_k$ such that

$$\int |u(z)|^2 \frac{1}{|\sin(\pi z)|^2} \frac{1}{(1+|z|^2)^3} < \infty.$$

Since $g \in L^\infty$, we also get

$$\int_{|\operatorname{Im}(z)| \geq 1} |h|^2 \frac{1}{|\sin(\pi z)|^2} \frac{1}{(1+|z|^2)^3} < \infty. \qquad \square$$

1.3 Plurisubharmonic functions

We want to prove L^2 estimates for the $\overline{\partial}$ problem in the case of several complex variables. We need to first say what the analogue of a subharmonic function is.

Definition 1.1. Let $\Omega \subseteq \mathbb{C}^n$ be open. A function $u : \Omega \to [-\infty, \infty)$ is called **plurisubharmonic** if

1. u is upper semicontinuous
2. for all $z \in \Omega$ and $w \in \mathbb{C}^n$, the function $\tau \mapsto u(z + \tau w)$ is subharmonic where it is defined.